10

Kronecker product, vec-operator, and Moore-Penrose inverse

The Kronecker product transforms two matrices $A := (a_{ij})$ and $B := (b_{st})$ into a matrix containing all products $a_{ij}b_{st}$. More precisely, let A be a matrix of order $m \times n$ and B a matrix of order $p \times q$. Then the $mp \times nq$ matrix defined by

$$\begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix}$$

is called the *Kronecker product* of A and B and is written $A \otimes B$. We notice that the Kronecker product $A \otimes B$ is defined for any pair of matrices A and B, irrespective of their orders.

The *vec-operator* transforms a matrix into a vector by stacking its columns one underneath the other. Let A be an $m \times n$ matrix and a_i its i-th column. Then $\operatorname{vec} A$ is the $mn \times 1$ vector

$$\operatorname{vec} A := \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Notice that $\operatorname{vec} A$ is defined for any matrix A, not just for square matrices. We shall see that the Kronecker product and the vec-operator are intimately connected. Also notice the notation. The expression $\operatorname{vec} A'$ denotes $\operatorname{vec}(A')$ and not $(\operatorname{vec} A)'$. Occasionally we shall use parentheses and write, for example, $\operatorname{vec}(AB)$ instead of $\operatorname{vec} AB$, but only if there is a possibility of confusion.

The inverse of a matrix is defined when the matrix is square and nonsingular. For many purposes it is useful to generalize the concept of invertibility to singular matrices and, indeed, to nonsquare matrices. One such generalization that is particularly useful because of its uniqueness is the *Moore-Penrose* (MP) inverse. Let $\boldsymbol{A}$ be a given real matrix of order $m \times n$. Then an $n \times m$ matrix $\boldsymbol{X}$ is said to be the MP-inverse of $\boldsymbol{A}$ if the following four conditions are satisfied:

$$\boldsymbol{AXA} = \boldsymbol{A}, \quad \boldsymbol{XAX} = \boldsymbol{X}, \quad (\boldsymbol{AX})' = \boldsymbol{AX}, \quad (\boldsymbol{XA})' = \boldsymbol{XA}.$$

We shall denote the MP-inverse of $\boldsymbol{A}$ by $\boldsymbol{A}^+$. Occasionally a more general inverse, the so-called *generalized inverse* $\boldsymbol{A}^-$, suffices. This matrix only satisfies one of the four equations, $\boldsymbol{AXA} = \boldsymbol{A}$, and it is not unique.

All matrices in this chapter are real, unless specified otherwise. However, most statements, for example the definition of the Moore-Penrose inverse above, generalize straightforwardly to complex matrices by replacing the transpose (′) by the conjugate transpose (*).

10.1 The Kronecker product

Exercise 10.1 (Kronecker examples) Let

$$\boldsymbol{A} = \begin{pmatrix} 2 & 5 & 2 \\ 0 & 6 & 3 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 2 & 4 & 1 \\ 3 & 5 & 0 \end{pmatrix}, \quad \boldsymbol{e}' = (0, 0, 1).$$

(a) Compute $\boldsymbol{I}_2 \otimes \boldsymbol{A}$ and $\boldsymbol{A} \otimes \boldsymbol{I}_2$.
(b) Compute $\boldsymbol{A}' \otimes \boldsymbol{B}$.
(c) Compute $\boldsymbol{A} \otimes \boldsymbol{e}$ and $\boldsymbol{A} \otimes \boldsymbol{e}'$.

Solution
(a) We have

$$\boldsymbol{I}_2 \otimes \boldsymbol{A} = \begin{pmatrix} \boldsymbol{A} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{A} \end{pmatrix} = \begin{pmatrix} 2 & 5 & 2 & 0 & 0 & 0 \\ 0 & 6 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 5 & 2 \\ 0 & 0 & 0 & 0 & 6 & 3 \end{pmatrix}$$

and

$$\boldsymbol{A} \otimes \boldsymbol{I}_2 = \begin{pmatrix} 2\boldsymbol{I}_2 & 5\boldsymbol{I}_2 & 2\boldsymbol{I}_2 \\ \mathbf{O} & 6\boldsymbol{I}_2 & 3\boldsymbol{I}_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 5 & 0 & 2 & 0 \\ 0 & 2 & 0 & 5 & 0 & 2 \\ 0 & 0 & 6 & 0 & 3 & 0 \\ 0 & 0 & 0 & 6 & 0 & 3 \end{pmatrix}.$$

(b) Similarly,

$$\boldsymbol{A}' \otimes \boldsymbol{B} = \begin{pmatrix} 2\boldsymbol{B} & \mathbf{O} \\ 5\boldsymbol{B} & 6\boldsymbol{B} \\ 2\boldsymbol{B} & 3\boldsymbol{B} \end{pmatrix} = \begin{pmatrix} 4 & 8 & 2 & 0 & 0 & 0 \\ 6 & 10 & 0 & 0 & 0 & 0 \\ 10 & 20 & 5 & 12 & 24 & 6 \\ 15 & 25 & 0 & 18 & 30 & 0 \\ 4 & 8 & 2 & 6 & 12 & 3 \\ 6 & 10 & 0 & 9 & 15 & 0 \end{pmatrix}.$$

(c) Finally,

$$A \otimes e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 5 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 6 & 3 \end{pmatrix}, \quad A \otimes e' = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 5 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 3 \end{pmatrix}.$$

Exercise 10.2 (Noncommutativity of Kronecker product) Show that:
(a) $\mathbf{O} \otimes A = A \otimes \mathbf{O} = \mathbf{O}$;
(b) $\mathrm{dg}(A \otimes B) = \mathrm{dg}(A) \otimes \mathrm{dg}(B)$ (A and B square);
(c) $A \otimes B \neq B \otimes A$, in general.

Solution
(a) The matrix $A \otimes B$ contains elements $a_{ij}b_{st}$. If either $a_{ij} = 0$ for all i, j or if $b_{st} = 0$ for all s, t, then $A \otimes B = \mathbf{O}$.
(b) Let A be a matrix of order $n \times n$. Then,

$$\mathrm{dg}(A \otimes B) = \mathrm{dg}\begin{pmatrix} a_{11}B & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & a_{22}B & \dots & \mathbf{O} \\ \vdots & \vdots & & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & a_{nn}B \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}\,\mathrm{dg}(B) & \mathbf{O} & \dots & \mathbf{O} \\ \mathbf{O} & a_{22}\,\mathrm{dg}(B) & \dots & \mathbf{O} \\ \vdots & \vdots & & \vdots \\ \mathbf{O} & \mathbf{O} & \dots & a_{nn}\,\mathrm{dg}(B) \end{pmatrix} = \mathrm{dg}(A) \otimes \mathrm{dg}(B).$$

(c) If A is of order $m \times n$ and B of order $p \times q$, then $A \otimes B$ and $B \otimes A$ have the same order $mp \times nq$ (in contrast to AB and BA, which may be of different orders). Exercise 10.1(a) contains an example of noncommutativity. Another example is

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}.$$

Exercise 10.3 (Kronecker rules) Show that the Kronecker product satisfies the following rules:
(a) $(A_1 + A_2) \otimes B = A_1 \otimes B + A_2 \otimes B$ (A_1 and A_2 of the same order);
(b) $A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2$ (B_1 and B_2 of the same order);
(c) $\alpha A \otimes \beta B = \alpha\beta(A \otimes B)$;
(d) $(A \otimes B)(C \otimes D) = AC \otimes BD$.

Solution

(a) The typical block in the matrix $\boldsymbol{A} \otimes \boldsymbol{B}$ is the submatrix $a_{ij}\boldsymbol{B}$. Hence, the typical block in $(\boldsymbol{A}_1 + \boldsymbol{A}_2) \otimes \boldsymbol{B}$ is

$$(\boldsymbol{A}_1 + \boldsymbol{A}_2)_{ij}\boldsymbol{B} = ((\boldsymbol{A}_1)_{ij} + (\boldsymbol{A}_2)_{ij})\boldsymbol{B} = (\boldsymbol{A}_1)_{ij}\boldsymbol{B} + (\boldsymbol{A}_2)_{ij}\boldsymbol{B}.$$

(b) Let $\boldsymbol{B} := \boldsymbol{B}_1 + \boldsymbol{B}_2$. Then for each submatrix $a_{ij}\boldsymbol{B}$ of $\boldsymbol{A} \otimes \boldsymbol{B}$, we have $a_{ij}\boldsymbol{B} = a_{ij}\boldsymbol{B}_1 + a_{ij}\boldsymbol{B}_2$.

(c) In each typical submatrix we have $(\alpha\boldsymbol{A})_{ij}(\beta\boldsymbol{B}) = (\alpha\beta)a_{ij}\boldsymbol{B}$.

(d) Let $\boldsymbol{A}$ $(m \times n)$, $\boldsymbol{B}$ $(q \times r)$, $\boldsymbol{C}$ $(n \times p)$, and $\boldsymbol{D}$ $(r \times s)$ be given matrices. Then both products $\boldsymbol{AC}$ and $\boldsymbol{BD}$ are defined. Since $\boldsymbol{A} \otimes \boldsymbol{B}$ has nr columns and $\boldsymbol{C} \otimes \boldsymbol{D}$ has nr rows, the product $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D})$ is defined as well. Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{C} = (c_{st})$. Then,

$$\begin{aligned}
(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) &= \begin{pmatrix} a_{11}\boldsymbol{B} & \dots & a_{1n}\boldsymbol{B} \\ \vdots & & \vdots \\ a_{m1}\boldsymbol{B} & \dots & a_{mn}\boldsymbol{B} \end{pmatrix} \begin{pmatrix} c_{11}\boldsymbol{D} & \dots & c_{1p}\boldsymbol{D} \\ \vdots & & \vdots \\ c_{n1}\boldsymbol{D} & \dots & c_{np}\boldsymbol{D} \end{pmatrix} \\
&= \begin{pmatrix} (\sum_i a_{1i}c_{i1})\boldsymbol{BD} & \dots & (\sum_i a_{1i}c_{ip})\boldsymbol{BD} \\ \vdots & & \vdots \\ (\sum_i a_{mi}c_{i1})\boldsymbol{BD} & \dots & (\sum_i a_{mi}c_{ip})\boldsymbol{BD} \end{pmatrix} \\
&= \begin{pmatrix} (\boldsymbol{AC})_{11}\boldsymbol{BD} & \dots & (\boldsymbol{AC})_{1p}\boldsymbol{BD} \\ \vdots & & \vdots \\ (\boldsymbol{AC})_{m1}\boldsymbol{BD} & \dots & (\boldsymbol{AC})_{mp}\boldsymbol{BD} \end{pmatrix} = \boldsymbol{AC} \otimes \boldsymbol{BD}.
\end{aligned}$$

Exercise 10.4 (Kronecker twice) Show that

$$\boldsymbol{A} \otimes (\boldsymbol{B} \otimes \boldsymbol{C}) = (\boldsymbol{A} \otimes \boldsymbol{B}) \otimes \boldsymbol{C},$$

and hence that $\boldsymbol{A} \otimes \boldsymbol{B} \otimes \boldsymbol{C}$ is unambiguous.

Solution

Let $\boldsymbol{A}$ be a matrix of order $m \times n$. Then,

$$\begin{aligned}
(\boldsymbol{A} \otimes \boldsymbol{B}) \otimes \boldsymbol{C} &= \begin{pmatrix} a_{11}\boldsymbol{B} & \dots & a_{1n}\boldsymbol{B} \\ \vdots & & \vdots \\ a_{m1}\boldsymbol{B} & \dots & a_{mn}\boldsymbol{B} \end{pmatrix} \otimes \boldsymbol{C} \\
&= \begin{pmatrix} (a_{11}\boldsymbol{B}) \otimes \boldsymbol{C} & \dots & (a_{1n}\boldsymbol{B}) \otimes \boldsymbol{C} \\ \vdots & & \vdots \\ (a_{m1}\boldsymbol{B}) \otimes \boldsymbol{C} & \dots & (a_{mn}\boldsymbol{B}) \otimes \boldsymbol{C} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}(\boldsymbol{B} \otimes \boldsymbol{C}) & \dots & a_{1n}(\boldsymbol{B} \otimes \boldsymbol{C}) \\ \vdots & & \vdots \\ a_{m1}(\boldsymbol{B} \otimes \boldsymbol{C}) & \dots & a_{mn}(\boldsymbol{B} \otimes \boldsymbol{C}) \end{pmatrix} = \boldsymbol{A} \otimes (\boldsymbol{B} \otimes \boldsymbol{C}).
\end{aligned}$$

Exercise 10.5 (Kroneckered by a scalar)
(a) Show, for any scalar α, that $\alpha \otimes \boldsymbol{A} = \alpha \boldsymbol{A} = \boldsymbol{A}\alpha = \boldsymbol{A} \otimes \alpha$.
(b) Hence, show that $(\boldsymbol{A} \otimes \boldsymbol{b})\boldsymbol{B} = (\boldsymbol{AB}) \otimes \boldsymbol{b}$ for any vector $\boldsymbol{b}$ (if $\boldsymbol{AB}$ is defined).

Solution
(a) This follows directly from the definition.
(b) In proofs, if at all possible, we wish to work with matrices and vectors, and not with the individual elements. So, we write

$$(\boldsymbol{A} \otimes \boldsymbol{b})\boldsymbol{B} = (\boldsymbol{A} \otimes \boldsymbol{b})(\boldsymbol{B} \otimes 1) = (\boldsymbol{AB}) \otimes \boldsymbol{b}.$$

(Check that all multiplications are allowed!)

Exercise 10.6 (Kronecker product of vectors) For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, not necessarily of the same order, show that $\boldsymbol{a} \otimes \boldsymbol{b}' = \boldsymbol{ab}' = \boldsymbol{b}' \otimes \boldsymbol{a}$.

Solution
Let $\boldsymbol{a} := (a_1, \dots, a_m)'$ and $\boldsymbol{b} := (b_1, \dots, b_n)'$. Then,

$$\boldsymbol{a} \otimes \boldsymbol{b}' = \begin{pmatrix} a_1\boldsymbol{b}' \\ \vdots \\ a_m\boldsymbol{b}' \end{pmatrix} = \boldsymbol{ab}' = (b_1\boldsymbol{a}, \dots, b_n\boldsymbol{a}) = \boldsymbol{b}' \otimes \boldsymbol{a}.$$

Exercise 10.7 (Transpose and trace of a Kronecker product) Show that:
(a) $(\boldsymbol{A} \otimes \boldsymbol{B})' = \boldsymbol{A}' \otimes \boldsymbol{B}'$;
(b) $\operatorname{tr}(\boldsymbol{A} \otimes \boldsymbol{B}) = (\operatorname{tr} \boldsymbol{A})(\operatorname{tr} \boldsymbol{B})$;
(c) $\boldsymbol{A} \otimes \boldsymbol{B}$ is idempotent if $\boldsymbol{A}$ and $\boldsymbol{B}$ are idempotent;
and specify the order conditions where appropriate.

Solution
(a) We have

$$(\boldsymbol{A} \otimes \boldsymbol{B})' = \begin{pmatrix} a_{11}\boldsymbol{B} & \dots & a_{1n}\boldsymbol{B} \\ \vdots & & \vdots \\ a_{m1}\boldsymbol{B} & \dots & a_{mn}\boldsymbol{B} \end{pmatrix}' = \begin{pmatrix} a_{11}\boldsymbol{B}' & \dots & a_{m1}\boldsymbol{B}' \\ \vdots & & \vdots \\ a_{1n}\boldsymbol{B}' & \dots & a_{mn}\boldsymbol{B}' \end{pmatrix} = \boldsymbol{A}' \otimes \boldsymbol{B}'.$$

This holds for matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of any order.
(b) Here, $\boldsymbol{A}$ and $\boldsymbol{B}$ must be square, but not necessarily of the same order. Then,

$$\begin{aligned} \operatorname{tr}(\boldsymbol{A} \otimes \boldsymbol{B}) &= \operatorname{tr} \begin{pmatrix} a_{11}\boldsymbol{B} & \dots & a_{1m}\boldsymbol{B} \\ \vdots & & \vdots \\ a_{m1}\boldsymbol{B} & \dots & a_{mm}\boldsymbol{B} \end{pmatrix} = \operatorname{tr}(a_{11}\boldsymbol{B}) + \dots + \operatorname{tr}(a_{mm}\boldsymbol{B}) \\ &= (a_{11} + \dots + a_{mm}) \operatorname{tr} \boldsymbol{B} = (\operatorname{tr} \boldsymbol{A})(\operatorname{tr} \boldsymbol{B}). \end{aligned}$$

(c) Again, A and B must be square, but not necessarily of the same order. Then,

$$(A \otimes B)(A \otimes B) = (AA) \otimes (BB) = A \otimes B.$$

Exercise 10.8 (Inverse of a Kronecker product) Show that

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

when A and B are both nonsingular, not necessarily of the same order.

Solution

Let A and B be nonsingular matrices of orders m and n, respectively. Then,

$$(A \otimes B)(A^{-1} \otimes B^{-1}) = (AA^{-1}) \otimes (BB^{-1}) = I_m \otimes I_n = I_{mn},$$

and hence one is the inverse of the other.

Exercise 10.9 (Kronecker product of a partitioned matrix) If A is a partitioned matrix,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

then show that

$$A \otimes B = \begin{pmatrix} A_{11} \otimes B & A_{12} \otimes B \\ A_{21} \otimes B & A_{22} \otimes B \end{pmatrix}.$$

Solution

Suppose A_{ij} has order $m_i \times n_j$, with $m_1 + m_2 = m$, $n_1 + n_2 = n$, so that the order of A is $m \times n$. Then,

$$A \otimes B = \begin{pmatrix} a_{1,1}B & \dots & a_{1,n_1}B & a_{1,n_1+1}B & \dots & a_{1,n}B \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m_1,1}B & \dots & a_{m_1,n_1}B & a_{m_1,n_1+1}B & \dots & a_{m_1,n}B \\ a_{m_1+1,1}B & \dots & a_{m_1+1,n_1} & a_{m_1+1,n_1+1}B & \dots & a_{m_1+1,n}B \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m,1}B & \dots & a_{m,n_1}B & a_{m,n_1+1}B & \dots & a_{m,n}B \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} \otimes B & A_{12} \otimes B \\ A_{21} \otimes B & A_{22} \otimes B \end{pmatrix}.$$

Exercise 10.10 (Eigenvalues of a Kronecker product) Let A be an $m \times m$ matrix with eigenvalues $\lambda_1, \dots, \lambda_m$, and let B be an $n \times n$ matrix with eigenvalues $\mu_1, \dots, \mu_n$. Show that the mn eigenvalues of $A \otimes B$ are $\lambda_i \mu_j (i = 1, \dots, m; j = 1, \dots, n)$.

Solution

By Schur's theorem (Exercise 7.62) there exist unitary matrices S and T such that

$$S^* AS = L, \quad T^* BT = M,$$

where L and M are upper triangular matrices whose diagonal elements are the eigenvalues of A and B, respectively. This gives

$$(S^* \otimes T^*)(A \otimes B)(S \otimes T) = L \otimes M.$$

Since $S^{-1} = S^*$ and $T^{-1} = T^*$, it follows that $(S \otimes T)^{-1} = S^* \otimes T^*$, and hence, using Exercise 7.24, that $(S^* \otimes T^*)(A \otimes B)(S \otimes T)$ and $A \otimes B$ have the same set of eigenvalues. This implies that $A \otimes B$ and $L \otimes M$ have the same set of eigenvalues. But $L \otimes M$ is an upper triangular matrix and hence (Exercise 7.15(b)) its eigenvalues are equal to its diagonal elements $\lambda_i \mu_j$.

Exercise 10.11 (Eigenvectors of a Kronecker product)
(a) Show that, if x is an eigenvector of A and y is an eigenvector of B, then $x \otimes y$ is an eigenvector of $A \otimes B$.
(b) Is it true that each eigenvector of $A \otimes B$ is of the form $x \otimes y$, where x is an eigenvector of A and y is an eigenvector of B?

Solution
(a) If $Ax = \lambda x$ and $By = \mu y$, then

$$(A \otimes B)(x \otimes y) = Ax \otimes By = \lambda x \otimes \mu y = \lambda\mu(x \otimes y).$$

(b) No. For example, let

$$A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Both eigenvalues of A (and both eigenvalues of B) are zero. The only eigenvector is e_1. The four eigenvalues of $A \otimes B$ are all zero, but $A \otimes B$ has three eigenvectors: $e_1 \otimes e_1$, $e_1 \otimes e_2$, and $e_2 \otimes e_1$.

Exercise 10.12 (Determinant and rank of a Kronecker product)
(a) If A and B are positive (semi)definite, show that $A \otimes B$ is positive (semi)definite.
(b) Show that

$$|A \otimes B| = |A|^n |B|^m,$$

when A and B are square matrices of orders m and n, respectively.
(c) Show that

$$\operatorname{rk}(A \otimes B) = \operatorname{rk}(A) \operatorname{rk}(B).$$

Solution
(a) If A and B are positive (semi)definite, then the eigenvalues λ_i of A and the eigenvalues μ_j of B are all positive (nonnegative), and hence all eigenvalues $\lambda_i \mu_j$ of $A \otimes B$ are also positive (nonnegative), so that $A \otimes B$ is positive (semi)definite (Exercise 8.11).

(b) The determinant is the product of the eigenvalues, so

$$|A \otimes B| = \prod_{i=1}^{m}\prod_{j=1}^{n}(\lambda_i \mu_j) = \prod_{i=1}^{m}(\lambda_i^n |B|) = |B|^m \left(\prod_{i=1}^{m}\lambda_i\right)^n = |A|^n |B|^m.$$

(c) Our starting point is

$$\mathrm{rk}(A \otimes B) = \mathrm{rk}(A \otimes B)(A \otimes B)' = \mathrm{rk}(AA' \otimes BB').$$

The matrix $AA' \otimes BB'$ is symmetric (in fact, positive semidefinite) and hence its rank equals the number of nonzero eigenvalues (Exercise 7.49). Now, the eigenvalues of $AA' \otimes BB'$ are $\{\lambda_i \mu_j\}$, where $\{\lambda_i\}$ are the eigenvalues of AA' and $\{\mu_j\}$ are the eigenvalues of BB'. The eigenvalue $\lambda_i \mu_j$ is nonzero if and only if both λ_i and μ_j are nonzero. Hence, the number of nonzero eigenvalues of $AA' \otimes BB'$ equals the product of the number of nonzero eigenvalues of AA' and the number of nonzero eigenvalues of BB'. This implies $\mathrm{rk}(A \otimes B) = \mathrm{rk}(A)\,\mathrm{rk}(B)$.

Exercise 10.13 (Nonsingularity of a Kronecker product)
If A and B are not square, then it is still possible that $A \otimes B$ is a square matrix. Show that $A \otimes B$ is singular, unless both A and B are square and nonsingular.

Solution
If the order of A is $m \times p$ and the order of B is $n \times q$, then $A \otimes B$ is square if and only if $mn = pq$. Now, by Exercises 10.12 and 4.7(a),

$$\mathrm{rk}(A \otimes B) = \mathrm{rk}(A)\,\mathrm{rk}(B) \leq \min(m, p)\min(n, q).$$

If $A \otimes B$ is nonsingular, we conclude from $\mathrm{rk}(A \otimes B) = mn$ that $m \leq p$, $n \leq q$, and from $\mathrm{rk}(A \otimes B) = pq$ that $p \leq m$ and $q \leq n$. Hence, $p = m$ and $q = n$, and A and B are both square. Since $\mathrm{rk}(A)\,\mathrm{rk}(B) = mn$, it follows that $\mathrm{rk}(A) = m$ and $\mathrm{rk}(B) = n$.

Exercise 10.14 (When is $A \otimes A \geq B \otimes B$?) If A and B are positive semidefinite, show that $A \otimes A \geq B \otimes B$ if and only if $A \geq B$.

Solution
If $A \geq B$, then

$$\begin{aligned} A \otimes A - B \otimes B &= A \otimes A - A \otimes B + A \otimes B - B \otimes B \\ &= A \otimes (A - B) + (A - B) \otimes B \geq O, \end{aligned}$$

since A, B, and $A - B$ are all positive semidefinite. Conversely, if $A \otimes A \geq B \otimes B$, then for any conformable x,

$$\begin{aligned} 0 \leq (x \otimes x)'(A \otimes A - B \otimes B)(x \otimes x) &= (x'Ax)^2 - (x'Bx)^2 \\ &= x'(A + B)x \cdot x'(A - B)x. \end{aligned}$$

Since $A \geq O$ and $B \geq O$, we have $x'(A+B)x \geq 0$. If $x'(A+B)x > 0$, then the above inequality implies that $x'(A-B)x \geq 0$. If $x'(A+B)x = 0$, then $x'Ax = x'Bx = 0$, so that $x'(A-B)x = 0$. It follows that $x'(A-B)x \geq 0$ for all x. Hence, $A \geq B$.

10.2 The vec-operator

Exercise 10.15 (Examples of vec) Compute vec A, vec A', vec B, vec B', vec e, vec e' of the matrices A and B, and the vector e in Exercise 10.1.

Solution
We have

$$\operatorname{vec} A = \begin{pmatrix} 2 \\ 0 \\ 5 \\ 6 \\ 2 \\ 3 \end{pmatrix}, \quad \operatorname{vec} A' = \begin{pmatrix} 2 \\ 5 \\ 2 \\ 0 \\ 6 \\ 3 \end{pmatrix}, \quad \operatorname{vec} B = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 1 \\ 0 \end{pmatrix}, \quad \operatorname{vec} B' = \begin{pmatrix} 2 \\ 4 \\ 1 \\ 3 \\ 5 \\ 0 \end{pmatrix},$$

and $\operatorname{vec} e = \operatorname{vec} e' = e = (0, 0, 1)'$.

Exercise 10.16 (Linearity of vec) If A and B have the same order, show that:
(a) $\operatorname{vec}(A + B) = \operatorname{vec}(A) + \operatorname{vec}(B)$;
(b) $\operatorname{vec}(\alpha A) = \alpha \operatorname{vec} A$.

Solution
(a) Let $A := (a_1, \dots, a_n)$, $B := (b_1, \dots, b_n)$, and $C := A + B$. Denoting the columns of C by $c_1, \dots, c_n$, we obtain

$$\operatorname{vec} C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \operatorname{vec} A + \operatorname{vec} B.$$

(b) Here,

$$\operatorname{vec}(\alpha A) = \begin{pmatrix} \alpha a_1 \\ \vdots \\ \alpha a_n \end{pmatrix} = \alpha \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \alpha \operatorname{vec} A.$$

Exercise 10.17 (Equality?)
(a) Does $\operatorname{vec} A = \operatorname{vec} B$ imply that $A = B$?
(b) Show that $\operatorname{vec} a' = \operatorname{vec} a = a$ for any vector a.